Batching and Optimal Multi-stage Bipartite Allocations

Yiding Feng (NWU) and Rad Niazadeh (UChicago)

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K-Stage Bipartite Matching

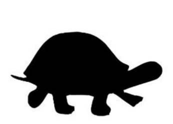
- Inspired by applications in matching market
- K-stage bipartite matching (with batch arrival)
  - offline supply nodes V, heterogeneous supply weight $\{w_j\}$
  - K stages, new batch U_k of demand nodes and their incident edges E_k arrive at stage $k \in [K]$
  - make irrevocable (fractional) matching decision s.t.

$$\sum_{j:(i,j)\in E_k} x_{ij} \le 1 \qquad i \in U_k$$
$$\sum_{i:(i,j)\in E_k} x_{ij} \le 1 - y_j \qquad j \in V$$
$$x_{ij} \ge 0 \qquad (i,j) \in E_k$$

where y_j is current consumption of supply node j

- Goal: maximize the competitive ratio (of matching weight $\sum w_j x_{ij}$) against optimum in-hindsight solution

$$\Gamma(K) \triangleq \sup_{\text{ALG}} \left(\inf_{G,\{w_j\}_{j\in V}} \frac{\mathbb{E}\left[\text{weight}(\text{ALG})\right]}{\text{weight}(\text{OPT-OFFLINE})} \right)$$



$K = 1$ (fully offline)	$K > 1$ (multi-stage)	$K = \lvert U \rvert$, batch size = 1 (fully online)

$\Gamma(1) = 1$ $\qquad\qquad$ $\Gamma(\infty) = 1 - \frac{1}{e}$

Formal question

In the middle-ground regime with $K > 1$ batches, what is <u>*optimal competitive ratio*</u> achieved by a multi-stage algorithm?

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Main Result: $\Gamma(K) = 1 - \left(1 - \frac{1}{K}\right)^K$

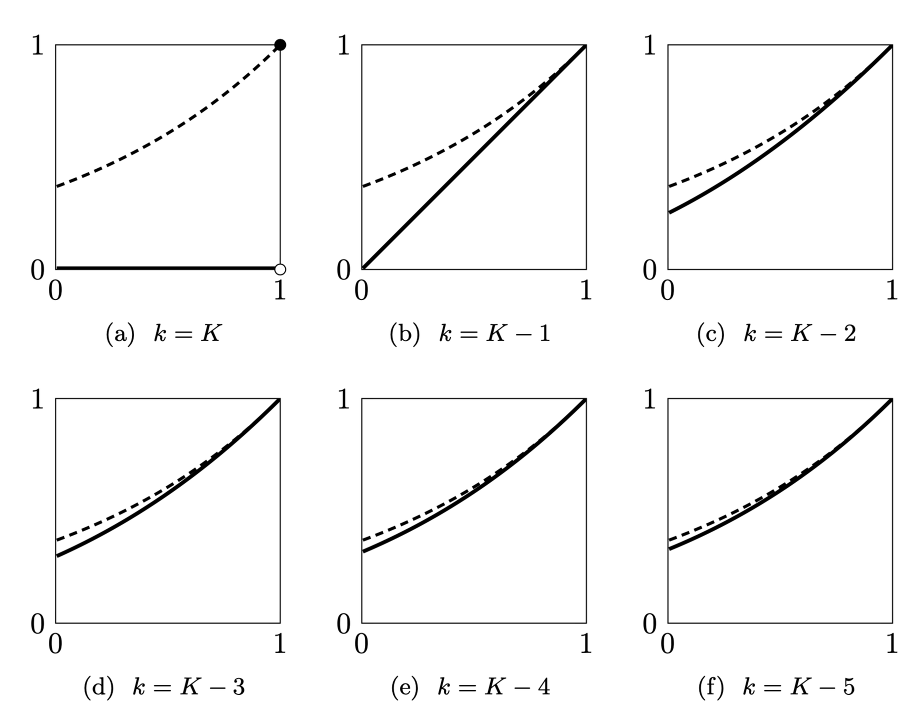
Thm. For K-stage fractional bipartite matching with any $K > 0$, we propose an efficient *(simple)* multi-stage algorithm with optimal competitive $\Gamma(K) = 1 - \left(1 - \frac{1}{K}\right)^K$.

We show same results hold when:

- *Fractional solution:* Vertex-weighted b-matching, budgeted allocation, assortment
- *Integral solution:* above settings, with the large budget assumption

Polynomial regularized convex-programming matching

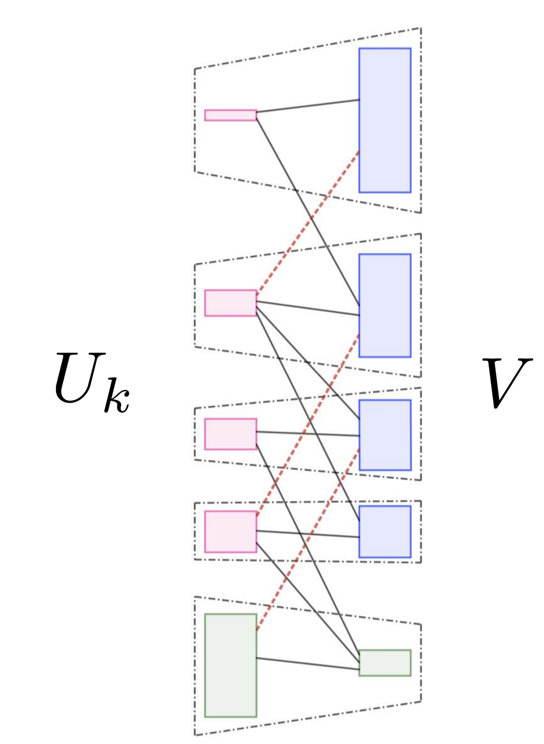
- Identifying a particular *sequence of polynomials of decreasing degrees* $\{f_k\}$, one for each batch
- Solving resulting *regularized max-weight matching convex programs* to find a family of balanced allocations
- Using *graph decompositions* induced by optimal solutions to come up with a new *primal-dual competitive analysis*



Sequence of polynomials

$$\operatorname*{argmax}_{\mathbf{x}} \sum_{(i,j)\in E_k} w_j x_{ij} - \sum_{j\in V} w_j F_k\Big(y_j + \sum_{i:(i,j)\in E_k} x_{ij}\Big)$$
$$\sum_{j:(i,j)\in E_k} x_{ij} \le 1 \qquad i \in U_k$$
$$\sum_{i:(i,j)\in E_k} x_{ij} \le 1 - y_j \qquad j \in V$$
$$x_{ij} \ge 0 \qquad (i,j) \in E_k$$

Regularized max-weight matching convex programs where $F_k(x) = \int_0^x f_k(t)dt$



Graph decomposition