Ethereum’s Transaction Fee Market Reform of EIP 1559 (Extended Abstract)

Barnabé Monnot[1], Hum Qing Ze[2], Koh Chuan Shen Marcus[2], Stefanos Leonardos[2], and Georgios Piliouras[2]

[1] Ethereum Foundation barnabe.monnot@ethereum.org
[2] Singapore University of Technology and Design, 8 Somapah Rd, Singapore 487372, Singapore
{stefanos_leonardos,georgios}@sutd.edu.sg,

Abstract. The scarcity of resources to process transactions on the Ethereum blockchain gives rise to a transaction fee market between miners and users. In a process which closely resembles a generalized first price auction, users bid the fees that they are willing to pay and miners decide which transactions to include. Along with the well-known inefficiencies of generalized first price auctions, users’ guesswork on the right fee causes unreasonable delays, extreme variability of fees even among transactions within a block and strategic behavior that ultimately threatens the stability of the blockchain.
In the present paper, we study Ethereum’s Improvement Proposal 1559 (EIP 1559) which aims to address these issues. At the core of the proposal lies a floating *basefee* which stipulates a minimum fee that each user needs to burn for their transaction to be processed. The basefee is dynamically adjusted between blocks to account for prevailing market conditions. Users’ bids comprise a maximum fee (*feecap*) and a maximum tip (*premium*) that they are willing to pay to the miner that will (eventually) include their transaction. Our preliminary results (both analytic and experimental) suggest that the basefee quickly settles to a stationary level. We test the robustness of this property under different market conditions (demand distributions) and assumptions on users’ strategic behavior and find that the proposed mechanism generally leads to a more efficient fee market that generates higher social welfare under various scenarios. We conclude with open questions and a roadmap for ongoing work.

Keywords: Ethereum · EIP 1559 · Transactions Fees · Strategic Users · Auctions.

1 Introduction

As the most widely adopted blockchain platform capable of running smart contracts, Ethereum has seen a steady growth in transactions, recently peaking at 1.7 million transactions within a day on 17th September 2020, with total collected fees of 42,763 ethers, at the price of 389.02 USD per ether, amounting to above $16.5 million in revenue.[3] These transactions are packed by miners that maintain the network into blocks. Every transaction requires a set of different operations, and each operation requires a predefined amount of gas to perform. Due to a potential high volume of transactions, miners will thus choose the transactions with the highest bid and the bidder pays for the inclusion of the transaction. These bids are in the form of gas prices, which is how much a user is willing to pay for each unit of gas used, and transactions will be chosen through a simple (generalized) first price auction mechanism. As the dynamics of interacting on the blockchain are dependent on this auction mechanism, this subjects Ethereum to the same well-documented problems of first price auctions ([18]) leading to volatile transaction fees, unnecessary transaction delays and instability of the blockchain when no block rewards are given and the blockchain relies solely on transaction fees [3].

To illustrate the detriments of a first price auction mechanism on the gas price, transaction data for a week from block 10,900,000 (timestamp Sep-20-2020 03:17:06 PM UTC) to block 10,942,000 (timestamp Sep-27-2020 02:40:14 AM UTC) was extracted. Plotting the cumulative distribution

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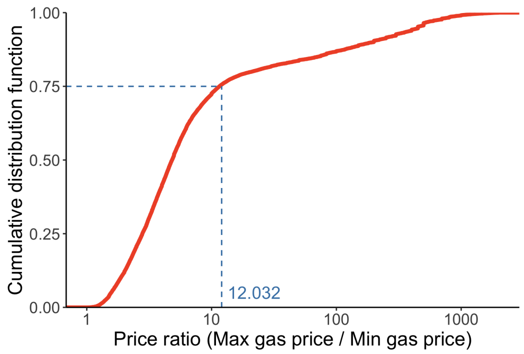
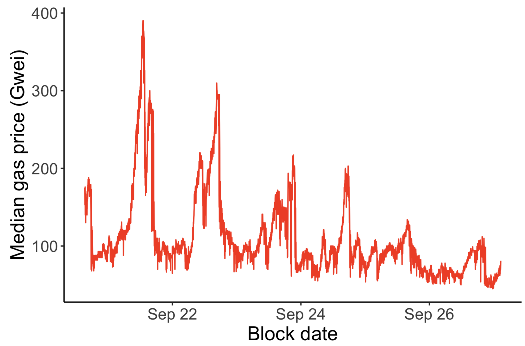
[3] Data collected from etherscan.io and coinmarketcap.com.

function of the ratio between the highest- and the lowest-paying transaction included in a block, as shown in Figure 1a, confirms the assumption of high variance in gas fees. It illustrates that over 25% of blocks report a price ratio above 12. To put this number in perspective, the situation is akin to paying at one petrol pump 12 times the price of the pump next to it, at the same time instant. The difficulty of estimating the correct gas price at the time of inclusion is compounded by the large fluctuations in median gas price, which went from 400 Gwei to less than 100 Gwei within the short time span of only a week (Figure 1b).



(a) Histogram of highest- to lowest-paying transaction included in a block.



(b) Median gas price in chunks of 10 blocks.

In order to solve the challenge of providing predictable gas prices,[4] EIP 1559 proposes the establishment of a minimum entry price, called *basefee*, for the gas price, which is adjusted based on network congestion. The basefee will then be burnt whenever a block is mined, and no participant will get to keep the value of the basefee. Users are free to propose a *miner premium*, added to the basefee, which is kept by the miner when including their transaction in the block. The premium incentivizes miners to process their transactions.

In our following preliminary results, we show that in the stationary demand case, basefee stabilises to some level determined by the distribution of user values and the miners' costs. This convergence is determined from a dynamical system obtained from the stochastic process governing the transaction fee market. Simulations (not included here) conducted with both strategic and non-strategic users suggest convergence of the basefee at the same stationary level. An additional experimental finding is that at equilibrium, the basefee reflects users' valuations of including their transaction in the block and no longer their urgency for inclusion.

2 The Model

Users submit transactions to a common pool. Along with their transaction, each user submits a bid, (f, p), where f denotes the maximum overall amount that the user is willing to expend for this transaction, called *feecap*, and p is the maximum tip that the user is willing to pay to the miner that will eventually process its transaction, called *miner premium*. Users have valuations, V, that are assumed to be independent and identically distributed (iid) with common distribution function $V \sim F$ with strictly positive support $S \subseteq \mathbb{R}_+$. For convenience, we assume throughout that F is continuous and strictly increasing. Whenever necessary, we will index users with $i, j \in \mathbb{N}$.

Any transaction that gets included in a block needs to expend (destroy or burn) a specific cost which is called *basefee*, $b > 0$, and a tip, τ, to the miner that will include the transaction. We will

[4] Note that EIP 1559 does not seek to address the *level* of gas prices, at least directly, which is determined by the underlying demand and supply pressure.

denote blocks with $B_t, t \geq 0$ where $t \in \mathbb{N}$ is the unique block height.[5] Thus, a transaction that gets included into block B_t expends a basefee b_t. Without loss of generality, we will assume that the time between blocks is constant and normalized to 1 time unit.

Miners view all transactions in the pool along with their bids and decide which transactions to include in the next block. We assume that miners are willing to process transactions only if their tip is at least some commonly known $\epsilon > 0$. Miners have an intrinsic marginal cost for including the transaction, which increases the size (in bytes) of the block and its propagation time over the network of miners, leading an increase in the probability of producing a stale block (called *uncle*). Thus, miners will select a transaction to be included in block B_t if $f > b_t + \epsilon$, and $p > \epsilon$, i.e., if the feecap is large enough to cover both the basefee and the minimum acceptable premium for miners, and the premium is large enough to satisfy the miners' tips. These conditions are summarized in the following minimum requirement

$$\min \{f - b_t, p\} > \epsilon. \tag{1}$$

Blocks have size T and a target block load $T/2$.[6] Let g_t denote the number of transactions that get included in block B_t. Since g_t depends on the basefee, b_t, we will write $g_t \mid b_t$ to denote the transactions that get included in B_t given that the basefee is equal to b_t. Then, the basefee of block B_{t+1} is updated via the following equation

$$b_{t+1} = b_t \left(1 + d \cdot \frac{g_t \mid b_t - T/2}{T/2}\right), \qquad \text{for any } t \in \mathbb{N}. \tag{2}$$

where d denotes an adjustment factor, currently set at $d = 0.125$. Equation (2) suggests that the basefee will increase if the load of block B_t is larger than the target block load, i.e., if there is increasing demand or congestion in the system, and otherwise will decrease. The magnitude of the change is regulated by the excess (shortage) of transaction load in the block compared to the target load and parameter d.

Our goal is to analyze the equilibrium behavior of the dynamical system that is determined by equation (2) under different assumptions regarding users' profiles (strategic behavior and characteristics).

3 Basefee equilibrium with nonstrategic users

In general, we will assume that transactions arrive to the pool according to a random process. Formally, let N_t denote the random number of transactions that arrive between two consecutive blocks B_t, B_{t+1} for $t \geq 0$. We assume that $N_t \sim \mathcal{P}(\lambda)$ for any $t \geq 0$, where $\mathcal{P}(\lambda)$ denotes the Poisson distribution of parameter λ. To avoid trivial cases, we will assume that $\lambda > T/2$, i.e., that the arrival rate is larger than the target block load.

To gain some intuition of the evolution of the basefee (i.e., of the long term behavior of the dynamical system in equation (2)), we start from a simple case in which users are nonstrategic. This means, that they bid $f = v$, i.e., the feecap is equal to their valuation, and $p = \epsilon$, i.e., the maximum tip to the miner who will (eventually) include their transaction is equal to the miners' acceptance threshold. Additionally, we will assume that users leave the pool if their transaction is not included in the next block.

Lemma 1 includes some first observations about the dynamic basefee. In particular, the basefee of block $t + 1$ depends only on the basefee of block t, which means that the *stochastic process*

[5] For the current analysis, we care only about blocks that make it into the *main* chain, and we will ignore orphaned blocks or forks.

[6] Size is measured in gas, i.e., typically T denotes the gas limit. Here, we express all measurements in units per gas, so under the assumption that all transactions use the same amount of gas, one may think of T as number of transactions.

$(b_t), t \geq 0$ has the *Markov property.*[7] This allows us to derive a closed form formula for the conditional expectation $\mathbb{E}[b_{t+1} \mid b_1, b_2, \dots, b_t]$ as given by equation (3).

Lemma 1. *Suppose that the number N_t of transactions that arrive to the transaction pool between consecutive blocks B_t, B_{t+1} follows a Poisson process with rate $\lambda \geq T/2$ for any $t \geq 0$ and that block size is unlimited. Further, suppose that users valuations $V_i, i = 1, \dots, N$ are iid with common distribution $V \sim F$ for some continuous and strictly increasing distribution function F and that users are nonstrategic, i.e., their bids satisfy $(f, p) = (V_i, \epsilon)$. Then, it holds that $b_t, t \geq 0$ has the Markov property and*

$$\mathbb{E}[b_{t+1} \mid b_t] = b_t + b_t \left(\frac{d}{T/2} \left[\lambda \left(1 - F(b_t + \epsilon)\right) - T/2 \right] \right). \tag{3}$$

Equation (3) in Lemma 1 suggests that the process $(b_t), t \geq 0$ does not satisfy a (sub/super)-martingale property. In particular, $\mathbb{E}[b_{t+1} \mid b_t]$ may be less than, equal to or larger than b_t depending on the sign of $\lambda(1 - F(b_t + \epsilon)) - T/2$. This complicates the convergence analysis of the stochastic process $(b_t), t \geq 0$.

To make some progress, we first study a closely related deterministic version of the above stochastic process. To formalize this statement, let $F^{-1}(p) := \inf\{x \in \mathbb{R} : F(x) \geq p\}$ denote the *inverse distribution function* of F. Since F is continuous and strictly increasing by assumption, it holds that for any $p \in [0, 1]$ there exists a unique $x \in \mathbb{R}$ such that $F^{-1}(p) = x$. Moreover, under these conditions, F^{-1} is also strictly increasing. Using this notation, we can prove Lemma 2.

Lemma 2. *Consider the deterministic process $b_t, t \geq 0$ with*

$$b_{t+1} = b_t + b_t \left(\frac{d}{T/2} \left[\lambda \left(1 - F(b_t + \epsilon)\right) - T/2 \right] \right).$$

Then, b_t has a unique stationary point given by

$$b^* = F^{-1}(1 - T/2\lambda) - \epsilon. \tag{4}$$

Moreover, b^* is attracting for any initial condition b_0 and $(b_t), t \geq 0$ converges to a db^*-neighborhood of b^*, i.e., there exists a $\bar{t} \in \mathbb{N}$, so that $b_t \in [(1 - d)b^*, (1 + d)b^*]$ for any $t > \bar{t}$.

Note that the deterministic dynamical system of Lemma 2 does not coincide with the actual stochastic system. However, it gives us some first intuition regarding the candidate equilibrium points of this system and the role of parameter d in the long-term (in-)stability of the basefee.

In the next section, we extend the study of $b_t, t \geq 0$ under gradual relaxation of the current assumptions and find empirical evidence that b^* plays an important role in the long-term evolution of the basefee even when the block size is limited and users behave strategically.

Remark 1. We used the assumption of unlimited block size to derive the expression in equation (6). Typically, the block size will be upper bounded by T (recall that $T/2$ is the target block load which is half of its total capacity). To study the conditional $\mathbb{E}[g_t \mid b_t]$ in equation (6) under the assumption

[7] Formally, a stochastic process $X_t, t \geq 0$ is *Markovian*, with respect to a filtration $\mathcal{F}_t = \sigma(X_s \mid s \leq t)$, if for any fixed time $t \geq 0$, the future of the process, i.e., X_{t+1}, is independent of $\mathcal{F}_t$ given X_t.

that the block size is equal to T, let for simplicity $P_T := P\left(\sum_{i=1}^N X_i > T\right)$. Then, we have

$$\begin{aligned}\mathbb{E}[g_t \mid b_t] &= \mathbb{E}\Big[\min\Big\{T, \sum_{i=1}^N X_i\Big\} \,\Big|\, b_t\Big] \\ &= T \cdot P\Big(\sum_{i=1}^N X_i > T\Big) + \mathbb{E}\Big[\sum_{i=1}^N X_i \,\Big|\, \sum_{i=1}^N X_i \le T\Big] \cdot P\Big(\sum_{i=1}^N X_i \le T\Big) \\ &= T \cdot P_T + \mathbb{E}\Big[\sum_{i=1}^N X_i \,\Big|\, \sum_{i=1}^N X_i \le T\Big] \cdot (1 - P_T).\end{aligned}$$

Although it captures reality more accurately, this expression is less tractable than (6). One way to proceed is to upper bound $\mathbb{E}[g_t \mid b_t]$ by interchanging the minimum with the expectation as follows.

$$\mathbb{E}[g_t \mid b_t] = \mathbb{E}\Big[\min\Big\{T, \sum_{i=1}^N X_i\Big\} \,\Big|\, b_t\Big] \le \min\Big\{T, \mathbb{E}\Big[\sum_{i=1}^N X_i \,\Big|\, b_t\Big]\Big\} = \min\{T, \lambda(1 - F(b_t + \epsilon))\}. \quad (5)$$

Plugging this into the expression for $\mathbb{E}[b_{t+1} \mid b_t]$ in the proof of Lemma 1 yields the inequality

$$\mathbb{E}[b_{t+1} \mid b_t] \le b_t \left(1 + \frac{d}{T/2}\left[\min\{T, \lambda\,(1 - F(b_t + \epsilon))\} - T/2\right]\right).$$

Again, each conditional expectations $\mathbb{E}[b_{t+1} \mid b_t]$ may be larger than, equal to or less than b_t and hence it is not straightforward to reason about the convergence of these sequence. Again, we may proceed as in Lemma 2, and consider the deterministic dynamical system

$$b_{t+1} = b_t \left(1 + \frac{d}{T/2}\left[\min\{T, \lambda\,(1 - F(b_t + \epsilon))\} - T/2\right]\right), \quad \text{for all } t \ge 0.$$

If for some $t \ge 0$, b_t is small enough so that $T > \lambda\,(1 - F(b_t + \epsilon))$, then $b_{t+1} = b_t(1 + d)$ which implies that $b_{t+1} > b_t$. As b_t grows larger, however, it will be the case that $F(b_t + \epsilon) \to 1$, i.e., $1 - F(b_t + \epsilon) \to 0$ which yields $\min\{T, \lambda(1 - F(b_t + \epsilon))\} = \lambda(1 - F(b_t + \epsilon))$. Thus, assuming that d is small enough, the long-term behavior of this system will be again determined by the term $\lambda(1 - F(b_t + \epsilon))$ and hence the system will converge to b^*.

4 Related Work

A healthy blockchain ecosystem requires interactions between miners that maintain the integrity of the blockchain and users that seek to carry out transactions on it. Protocols need to consider multiple properties that can affect the stability of the system regardless of the number of agents or behaviors [6,12]. When Bitcoin was examined as an equilibrium model of exogenously specified transactions fees with the block size assumed restricted to a single transaction, it predicts that miners' profits are zero and that fees are positively correlated with transaction waiting times [9]. This is further supported by work demonstrating that expected decreases in the production of blocks will lead to greater price volatility of the cryptocurrency [20]. A fee based on the average revenue generated over previous blocks was shown to reduce the variance of mining income by an average of 7.4 times [2], lending some credence to the idea that fees might help improve volatility. This protocol also contributes to the robustness of the blockchain by mitigating fee manipulation by miners. It reminds mechanism designers that changes to the protocol needs to also mitigate the security and performance issues that arise from the inevitable switch to a purely fee-based blockchain protocol without a block reward [4].

Other works attempt to improve Bitcoin's efficiency by modelling the system as a decentralized payment processor with many firms [10]. Other methods that prioritize revenue-maximization through changing the size of the block have also been proposed [8]. Empirical studies of Bitcoin have shown that the level of transaction fees may be influenced by social norms and conventions by key actors of the ecosystem such as powerful mining pools rather than the implied market mechanism [17]. Allowing users to capitalize on platform growth results in a intertemporal feedback between user adoption and token price which accelerates adoption and dampens user-base volatility [7]. However, this centralization of power might be detrimental to the overall ecosystem in the long run [1,11].

Ethereum, the blockchain of interest for this paper, does more than include transactions in blocks. The ability to carry out computation on the blockchain is priced through gas fees and thus will need to be considered when carrying out mining activity. A more complex approach proposes a composite fee taking into account the different resources spent by miners resulting in a state-related fee in the form of regular payments to miners[5]. The goal of such an approach was to prevent spam, manage storage demands, and lower network load. It was shown that the biggest factors influencing Ethereum's transaction fees are the number of pending transactions and the number of miners, implying a similar result to that of Bitcoin as these two factors affect waiting times for transactions the most. [19]. Indeed the concerns of managing transaction fees have led to a few attempts to reduce its volatility and EIP 1559 is the latest such proposal building on the work of others.

5 Open Questions and Ongoing Work

The current abstract and the referenced materials in [13,14,15,16] paint only a first picture of the properties of the basefee that is introduced in EIP 1559. While the updates of the basefee form a well-defined (stochastic) process, cf. equation (2), the convergence properties of this process are still not well understood. Under the assumption that arrivals (of transactions) follow a Poisson process with constant rate, simulations show that the basefee indeed quickly stabilizes around some the quantile of the value distribution (the quantile after which top users fill the block to its target size). Interestingly, this stability result seems to hold in the presence of transaction pool (rather than the naive assumption that we used here that users leave immediately if their transaction is not included in the first block after they arrive) and even in the presence of strategic users (users who bid above ϵ when blocks become full).

Overall, if these simulation results can be generalized, they would not only show that there is some robustness to the basefee (transaction fee price) given i.i.d. conditions, but also provide a measure of how fast the price converges, depending on d, where d is the control parameter of the basefee update. However, beyond some initial progress, obtaining general analytic results remains still elusive to us. Current questions that we are working on concern different types of convergence (for instance, almost sure convergence seems too strong, but it is not clear if one can establish convergence of this stochastic process in probability, in expectation or in some other weaker form). In any case, this is an actively researched topic, not only academically but primarily within the Ethereum community, and grasping the long term behavior of the proposed mechanism in EIP 1559 would be a significant step forward.

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A Omitted Proofs

Proof of Lemma 1. The Markovian property is immediate from equation (2) since b_{t+1} is fully determined by b_t and $g_t \mid b_t$. Thus, $\mathbb{E}[b_{t+1} \mid b_0, \dots, b_t] = \mathbb{E}[b_{t+1} \mid b_t]$ for any $t \geq 0$. To analyze the resulting dynamical system $b_t, t \geq 0$, we define the random variables

$$X_i = \begin{cases} 1, & \text{if } i\text{'s transaction gets included in block } B_t, \\ 0, & \text{otherwise.} \end{cases}$$

Since users bid $(f, p) = (V_i, \epsilon)$ (by the assumption that they are nonstrategic), it holds that $X_i = \mathbf{1}\{V_i > b_t + \epsilon\}$. Moreover,

$$P(X_i = 1) = P(V_i > b_t + \epsilon) = 1 - F(b_t + \epsilon), \quad \text{for any } i = 1, 2, \dots, N.$$

Thus, the expected value of $g_t \mid b_t$ is equal to

$$\mathbb{E}[g_t \mid b_t] = \mathbb{E}\Big[\sum_{i=1}^{N} X_i \;\Big|\; b_t\Big] = \mathbb{E}[N]\mathbb{E}[X \mid b_t] = \lambda \cdot (1 - F(b_t + \epsilon)), \tag{6}$$

since the random variables X_i are independent of N (we omitted the subscript t for simplicity). Taking the conditional expectation of equation (2) and substituting the above expression yields the result in equation (3). Specifically,

$$\mathbb{E}[b_{t+1} \mid b_t] = \mathbb{E}\left[b_t \left(1 + d \cdot \frac{g_t \mid b_t - T/2}{T/2}\right) \,\middle|\, b_t\right] = b_t \left(1 + d \cdot \frac{\mathbb{E}[g_t \mid b_t] - T/2}{T/2}\right)$$
$$= b_t + b_t \left(\frac{d}{T/2}\left[\lambda\left(1 - F(b_t + \epsilon)\right) - T/2\right]\right). \qquad \square$$

Proof of Lemma 2. Let $r_t := \frac{d}{T/2} \cdot [\lambda(1 - F(b_t + \epsilon)) - T/2]$ denote the *rate of change* of b_t. The process b_t becomes stationary if $r_t = 0$, i.e., if $\lambda(1 - F(b_t + \epsilon)) - T/2 = 0$, which we can solve for b_t to obtain a unique solution

$$b^* = F^{-1}(1 - T/2\lambda) - \epsilon.$$

Hence, b^* is the only candidate equilibrium (fixed or rest point) of the dynamical system $b_t, t \geq 0$. To see that the sequence b_t will indeed move towards b^*, it suffices to observe that b^* is *stable* or *attracting*. Since the dynamical system is one-dimensional, this follows from a sign analysis of r_t. If $b_t > b^*$, then $F(b_t + \epsilon) > F(b^* + \epsilon)$ which implies that $r_t < 0$. Similarly, whenever $b_t < b^*$, it follows that $r_t > 0$. However, unless $b_t = b^*$ for some $t \geq 0$, the system will only move in a neighborhood of b^* which depends on d. In any case, this neighborhood cannot be larger than db^* (when the system goes from a value above b^* to a value below b^* and vice versa), which concludes the proof. $\square$